

BGG CORRESPONDENCE AND RÖMER'S THEOREM ON AN EXTERIOR ALGEBRA

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To the memory of Professor Tetsushi Ogoma

ABSTRACT. Let $E = K\langle y_1, \dots, y_n \rangle$ be the exterior algebra. The (*cohomological distinguished pairs*) of a graded E -module N describe the growth of a minimal graded injective resolution of N . Römer gave a duality theorem between the distinguished pairs of N and those of its dual N^* . In this paper, we show that under Bernstein-Gel'fand-Gel'fand correspondence, his theorem is translated into a natural corollary of local duality for (complexes of) graded $S = K[x_1, \dots, x_n]$ -modules. Using this idea, we also give a \mathbb{Z}^n -graded version of Römer's theorem.

INTRODUCTION

In this section, to introduce a background of the present paper, we summarize results of Aramova-Herzog [2] and Römer [11].

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K , and M a finitely generated graded S -module. The ij th Betti number $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^S(K, M)_j$ of M is an important invariant. Following Bayer-Charalambous-Popescu [3], we say a Betti number $\beta_{k,m}(M) \neq 0$ is *extremal*, if $\beta_{i,j}(M) = 0$ for all $(i, j) \neq (k, m)$ with $i \geq k$ and $j - i \geq m - k$. This notion has two remarkable properties. First, a homogeneous ideal $I \subset S$ has the same extremal Betti numbers as its generic initial ideal $\operatorname{Gin}(I)$ with respect to the reverse lexicographic term order on S . Another important property is the following.

Theorem A (Bayer-Charalambous-Popescu, [3, Theorem 2.8]) *Let $\Delta \subset 2^{\{1, \dots, n\}}$ be a simplicial complex, and $K[\Delta] = S/I_\Delta$ the Stanley-Reisner ring. And let Δ^\vee be the Alexander dual complex of Δ . Then $\beta_{i,i+j}(K[\Delta])$ is extremal if and only if so is $\beta_{j,i+j}(I_{\Delta^\vee})$. Moreover, if this is the case, then $\beta_{i,i+j}(K[\Delta]) = \beta_{j,i+j}(I_{\Delta^\vee})$.*

We have $\beta_{i,n}(K[\Delta]) = \dim_K \tilde{H}_{n-i-1}(|\Delta|; K) =: \tilde{h}_{n-i-1}(|\Delta|)$ by Hochster's formula. If $\beta_{i,n}(K[\Delta]) \neq 0$ then it is always an extremal Betti number. The equality $\tilde{h}_{n-i-1}(|\Delta|) = \beta_{i,n}(K[\Delta]) = \beta_{n-i,n}(I_{\Delta^\vee}) = \beta_{n-i+1,n}(K[\Delta^\vee]) = \tilde{h}_{i-2}(|\Delta^\vee|)$ induced by Theorem A corresponds to the usual Alexander duality. More generally, Theorem A gives an Alexander duality for (some) *iterated Betti numbers* (c.f. [9, 4]).

Let $E = K\langle y_1, \dots, y_n \rangle$ be the exterior algebra. To understand Theorem A, Aramova-Herzog [2] introduced *distinguished pairs* for a graded E -module N . See Definition 1.6 below. (We use a different convention to describe these pairs. See Remark 1.7.) The distinguished pairs of N roughly describe the growth of the minimal graded (infinite) injective resolution of N . Let $K\{\Delta\} = E/J_\Delta$ be the

exterior face ring of Δ . Then [2, Corollary 9.6] states that (d, i) is a distinguished pair for $K\{\Delta\}^* := \text{Hom}_E(K\{\Delta\}, E)$ if and only if $\beta_{d+i-n, d}(K[\Delta])$ is extremal. And the values of extremal Betti numbers can be described by the Cartan cohomologies of $K\{\Delta\}^*$.

Römer proved that (d, i) is distinguished for N if and only if so is $(d, 2n - d - i)$ for N^* . He also gave a duality between certain components of Cartan cohomologies of N and those of N^* . Since $k\{\Delta\}^* = J_{\Delta^\vee}$, his result implies Theorem A.

Bernstein-Gel'fand-Gel'fand correspondence (BGG correspondence, for short) is a well known theorem which states that the derived category $D^b(\text{gr } S)$ of finitely generated graded S -modules is equivalent to the similar category $D^b(\text{gr } E)$ for E . In this paper, we give a new proof of the result of Römer using BGG correspondence. More precisely, under this correspondence, Römer's theorem is translated into a statement on $D^b(\text{gr } S)$ which is a natural consequence of the local duality (Serre duality). A key point is that the duality functor $\text{Hom}_E(-, E)$ on $D^b(\text{gr } E)$ corresponds to the duality functor $R\text{Hom}_S(-, \omega^\bullet)$ on $D^b(\text{gr } S)$, where ω^\bullet is a dualizing complex of S . So our proof is (philosophically) simple and sheds new light on the result.

The original paper [3] states Theorem A in the \mathbb{Z}^n -graded context, while the arguments in [2, 11] are hard to work in this context. But, since BGG correspondence also holds for \mathbb{Z}^n -graded modules, our method is powerful in this context too. See §2. This part of the present paper is a continuation of the author's previous paper [13].

1. \mathbb{Z} -GRADED CASE

Let W be an n -dimensional vector space over a field K , and $S = \bigoplus_{i \geq 0} \text{Sym}_i W$ the polynomial ring. We regard S as a graded ring with $S_i = \text{Sym}_i W$. Let $\text{Gr } S$ be the category of graded S -modules and their degree preserving S -homomorphisms, and $\text{gr } S$ the full subcategory of $\text{Gr } S$ consisting of finitely generated modules. Then there is an equivalence $D^b(\text{gr } S) \cong D^b_{\text{gr } S}(\text{Gr } S)$. (For derived categories, consult [8].) So we will freely identify these categories. For $M = \bigoplus_{i \in \mathbb{Z}} M_i \in \text{Gr } S$ and an integer j , $M(j)$ denotes the shifted module with $M(j)_i = M_{i+j}$. For $M^\bullet \in D^b(\text{Gr } S)$, $M^\bullet[j]$ denotes the j th translation of M^\bullet , that is, $M^\bullet[j]$ is the complex with $M^\bullet[j]^i = M^{i+j}$. So, if $M \in \text{Gr } S$, $M[j]$ is the cochain complex $\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$, where M sits in the $(-j)$ th position. If $M \in \text{gr } S$ and $N \in \text{Gr } S$, then $\text{Hom}_S(M, N)$ has the structure of a graded S -module with $\text{Hom}_S(M, N)_i = \text{Hom}_{\text{Gr } S}(M, N(i))$.

Let $\omega^\bullet \in D^b(\text{gr } S)$ be a minimal graded injective resolution of $S(-n)[n]$. That is, ω^\bullet is a graded normalized dualizing complex of S . Then $\mathbf{D}_S(-) := \text{Hom}_S^\bullet(-, \omega^\bullet)$ gives a duality functor from $D^b(\text{gr } S)$ to itself. The i th cohomology of $\mathbf{D}_S(M^\bullet)$ is $\text{Ext}_S^i(M^\bullet, \omega^\bullet)$. For $M^\bullet \in D^b(\text{gr } S)$ and $i \in \mathbb{Z}$, set $d_i(M^\bullet) := \dim_S H^i(M^\bullet)$. Here the Krull dimension of the 0 module is $-\infty$.

Definition 1.1. We say $(d, i) \in \mathbb{N} \times \mathbb{Z}$ is a *distinguished pair* for a complex $M^\bullet \in D^b(\text{gr } S)$, if $d = d_i(M^\bullet)$ and $d_j(M^\bullet) < d + i - j$ for all j with $j < i$.

Let $M^\bullet \in D^b(\text{gr } S)$ and $d = d_i(M^\bullet) \geq 0$. If $d = \max\{d_j(M^\bullet) \mid j \in \mathbb{Z}\}$, then (d, i) is distinguished for M^\bullet . On the other hand, if $i = \min\{j \mid H^j(M^\bullet) \neq 0\}$, then (d, i) is also distinguished. Thus M^\bullet has several distinguished pairs in general.

In this paper, $\deg_S(M)$ denotes the multiplicity of a module $M \in \text{gr } S$ (i.e., $e(M)$ of [5, Definition 4.1.5]).

Theorem 1.2. *For $M^\bullet \in D^b(\text{gr } S)$, we have the following.*

(1) *A pair (d, i) is distinguished for M^\bullet if and only if $(d, -d - i)$ is distinguished for $\mathbf{D}_S(M^\bullet)$.*

(2) *If (d, i) is a distinguished pair for M^\bullet , then*

$$\deg_S H^i(M^\bullet) = \deg_S \text{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet).$$

Proof. (1) Since the statement is “symmetric”, it suffices to prove the direction \Rightarrow .

From the double complex $\text{Hom}_S^\bullet(M^\bullet, \omega^\bullet)$, we have a spectral sequence $E_2^{p,q} = \text{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \text{Ext}_S^{p+q}(M^\bullet, \omega^\bullet)$. For simplicity, set $e_r^{p,q} := \dim_S E_r^{p,q}$. Since $\text{Ext}_S^i(M, \omega^\bullet) \cong \text{Ext}_S^{n+i}(M, S(-n))$ for $M \in \text{gr } S$, the following inequality follows from argument analogous to [5, §8.1, Theorem 8.1.1].

$$(1.1) \quad e_2^{p,q} = \dim_S \text{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) = \begin{cases} -p & \text{if } p = -d_{-q}(M^\bullet), \\ \leq -p & \text{if } -d_{-q}(M^\bullet) < p \leq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

(I) By (1.1), we have $e_2^{-d,-i} = d$. On the other hand, we have $e_2^{p,q} < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$. In fact, the assertion follows from (1.1) if $p > -d$. So we may assume that $p < -d$ and $q = -d - i - p > -i$. Since (d, i) is distinguished, we have $d_{-q}(M^\bullet) < d + i + q = -p$. Thus $E_2^{p,q} = 0$ in this case. Anyway, we have $e_\infty^{p,q} < d$ for all $(p, q) \neq (-d, -i)$ with $p + q = -d - i$.

(II) Since $d_{i-j+1}(M^\bullet) < d + j - 1 < d + j$ for all $j \geq 2$, we have that $E_2^{-d-j, -i+j-1} = 0$. So we have $E_r^{-d-j, -i+j-1} = 0$ for all $r \geq 2$. Next we will show that $d = e_2^{-d,-i} = e_3^{-d,-i} = \dots = e_r^{-d,-i}$ by induction on r . Recall that $E_{r+1}^{-d,-i}$ is the cohomology of

$$E_r^{-d-r, -i+r-1} \rightarrow E_r^{-d,-i} \rightarrow E_r^{-d+r, -i-r+1}.$$

But we have seen that $E_r^{-d-r, -i+r-1} = 0$. Moreover, $e_r^{-d+r, -i-r+1} \leq e_2^{-d+r, -i-r+1} \leq d - r < d$ by (1.1), and $e_r^{-d,-i} = d$ by the induction hypothesis. Thus $e_{r+1}^{-d,-i} = d$. Hence $e_\infty^{-d,-i} = d$. From this fact and (I), we have that $\dim_S \text{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet) = d$.

(III) Finally, we will show that $\dim_S \text{Ext}_S^{-d-i-j}(M^\bullet, \omega^\bullet) < d + j$ for all $j > 0$. To see this, it suffices to show that $e_2^{p,q} < d + j$ for all $j > 0$ and all (p, q) with $p + q = -d - i - j$. If $p > -d - j$, the assertion is clear. If $p = -d - j$, then $q = -i$ and $d_{-q}(M^\bullet) = d < -p$. So $E_2^{p,q} = 0$ in this case. Hence we may assume that $p < -d - j$ and $-q = d + i + j + p < i$. Since (d, i) is distinguished, $d_{-q}(M^\bullet) < d + (i + q) = -j - p < -p$. So we have $E_2^{p,q} = 0$ in this case too.

(2) Since $\deg_S E_r^{-d,-i} = \deg_S E_{r+1}^{-d,-i}$ for all $r \geq 2$ by the argument in (II) of the proof of (1), we have $\deg_S E_2^{-d,-i} = \deg_S E_\infty^{-d,-i}$. So we have

$$\deg_S \text{Ext}_S^{-d-i}(M^\bullet, \omega^\bullet) = \deg_S E_\infty^{-d,-i} = \deg_S E_2^{-d,-i} = \deg_S \text{Ext}_S^{-d}(H^i(M^\bullet), \omega^\bullet),$$

where the first equality follows from (I) and (II).

For a module $M \in \text{gr } S$ of dimension d , we have $\deg_S M = \deg_S \text{Ext}_S^{-d}(M, \omega^\bullet)$. In fact, for a prime ideal $\mathfrak{p} \subset S$ with $\dim S/\mathfrak{p} = d$, let \mathbf{x} be a maximal $S_{\mathfrak{p}}$ -sequence contained in $\text{Ann}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}})$, and $R := S_{\mathfrak{p}}/\mathbf{x}S_{\mathfrak{p}}$ an artinian Gorenstein local ring. Then we have

$$\text{Ext}_S^{-d}(M, \omega^\bullet) \otimes_S S_{\mathfrak{p}} \cong \text{Ext}_{S_{\mathfrak{p}}}^{n-d}(M_{\mathfrak{p}}, S_{\mathfrak{p}}) \cong \text{Hom}_R(M_{\mathfrak{p}}, R).$$

Therefore,

$$l_{S_{\mathfrak{p}}}(\text{Ext}_S^{-d}(M, \omega^\bullet) \otimes_S S_{\mathfrak{p}}) = l_R(\text{Hom}_R(M_{\mathfrak{p}}, R)) = l_R(M_{\mathfrak{p}}) = l_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since $\dim_S(H^i(M^\bullet)) = d$, we have $\deg_S \text{Ext}_S^{-d}(H^i(M), \omega^\bullet) = \deg_S H^i(M^\bullet)$. \square

Remark 1.3. Theorem 1.2 (1) also holds for a noetherian local ring A admitting a dualizing complex. The part (2) also holds for A , if we replace $\deg_S(-)$ by $l_{A_{\mathfrak{p}}}(- \otimes_A A_{\mathfrak{p}})$ for a prime ideal $\mathfrak{p} \subset A$ with $\dim A/\mathfrak{p} = d$.

Let V be the dual vector space of W , and $E = \bigwedge V$ the exterior algebra. We regard E as a negatively graded ring with $E_{-i} = \bigwedge^i V$ (this is the opposite convention from [2, 11]). Let $\text{gr } E$ be the category of finitely generated graded E -modules and their degree preserving E -homomorphisms. Here “ E -module” means a left and right module N with $ea = (-1)^{(\deg e)(\deg a)}ae$ for all homogeneous elements $e \in E$ and $a \in N$.

Let $\{x_1, \dots, x_n\}$ be a basis of W , and $\{y_1, \dots, y_n\}$ its dual basis of V . For a complex N^\bullet in $\text{gr } E$, set $\mathbf{L}(N^\bullet) = \bigoplus_{i \in \mathbb{Z}} S \otimes_K N^i$ and $\mathbf{L}(N^\bullet)^m = \bigoplus_{i-j=m} S \otimes_K N_j^i$. The differential defined by

$$\mathbf{L}(N^\bullet)^m \supset S \otimes_K N_j^i \ni 1 \otimes z \mapsto \sum_{1 \leq l \leq n} x_l \otimes y_l z + (-1)^m (1 \otimes \delta^i(z)) \in \mathbf{L}(N^\bullet)^{m+1}$$

makes $\mathbf{L}(N^\bullet)$ a cochain complex of free S -modules. Here δ^i is the i th differential map of N^\bullet . Moreover, \mathbf{L} gives a functor from $D^b(\text{gr } E)$ to $D^b(\text{gr } S)$.

For $M \in \text{gr } S$ and $i \in \mathbb{Z}$, we can define a graded E -module structure on $\text{Hom}_K(E, M_i)$ by $(af)(e) = f(ea)$. Then $\text{Hom}_K(E, M_i) \cong E(-n) \otimes_K M_i$. Set $\mathbf{R}(M) = \text{Hom}_K(E, M)$ and $\mathbf{R}^i(M) = \text{Hom}_K(E, M_i)$. The differential defined by

$$\mathbf{R}^i(M) = \text{Hom}_K(E, M_i) \ni f \mapsto [e \mapsto \sum_{1 \leq j \leq n} x_j f(y_j e)] \in \text{Hom}_K(E, M_{i+1}) = \mathbf{R}^{i+1}(M)$$

makes $\mathbf{R}(M)$ a cochain complex of free E -modules. We can also construct $\mathbf{R}(M^\bullet)$ from a complex M^\bullet in natural way. Then \mathbf{R} gives a functor from $D^b(\text{gr } S)$ to $D^b(\text{gr } E)$. See [6] for details. The following is a crucial result.

Theorem 1.4 (BGG correspondence, c.f.[6]). *The functors \mathbf{L} and \mathbf{R} give a category equivalence $D^b(\text{gr } S) \cong D^b(\text{gr } E)$.*

For $N \in \text{gr } E$, then $N^* := \text{Hom}_E(N, E) \cong \text{Hom}_K(N, K)(n)$ is a graded E -module again. $(-)^*$ gives an exact duality functor on $\text{gr } E$, and it can be extended to the duality functor \mathbf{D}_E on $D^b(\text{gr } E)$.

Proposition 1.5. *For $N^\bullet \in D^b(\text{gr } E)$, we have*

$$\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n].$$

Proof. Since $\mathbf{L}(N^\bullet)$ consists of free S -modules, we have

$$\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \text{Hom}_S^\bullet(\mathbf{L}(N^\bullet), S(-n)[n]).$$

It is easy to see that

$$\text{Hom}_S^m(\mathbf{L}(N^\bullet), S(-n)[n]) \cong \bigoplus_{j-i=m+n} S(-n) \otimes_K (N_j^i)^\vee,$$

where $(-)^\vee$ means the graded K -dual. On the other hand,

$$\begin{aligned} \mathbf{L} \circ \mathbf{D}_E(N^\bullet)^m &= \bigoplus_{i-j=m} S \otimes_K \mathbf{D}_E(N^\bullet)_j^i = \bigoplus_{i-j=m} S(n) \otimes_K (N_{-n-j}^{-i})^\vee \\ &= \bigoplus_{j-i=m-n} S(n) \otimes_K (N_j^i)^\vee. \end{aligned}$$

So we can easily construct a quasi-isomorphism $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \rightarrow \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n]$. \square

For $N^\bullet \in D^b(\text{gr } E)$, we have $H^i(\mathbf{L}(N^\bullet))_j \cong \text{Ext}_E^{j+i}(K, N^\bullet)_j$ by [6, Theorem 3.7]. So the Laurent series $P_i(t) = \sum_{j \in \mathbb{Z}} (\dim_K \text{Ext}_E^{j+i}(K, N^\bullet)_j) \cdot t^j$ is the Hilbert series of the finitely generated graded S -module $H^i(\mathbf{L}(N^\bullet))$. If $H^i(\mathbf{L}(N^\bullet)) \neq 0$, there exists a Laurent polynomial $Q_i(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$P_i(t) = \frac{Q_i(t)}{(1-t)^d},$$

where $d = d_i(\mathbf{L}(N^\bullet)) = \dim_S H^i(\mathbf{L}(N^\bullet))$. Set $e_i(N^\bullet) := Q_i(1) = \deg_S H^i(\mathbf{L}(N^\bullet))$. So $d_i(\mathbf{L}(N^\bullet))$ and $e_i(N^\bullet)$ measure the growth of the “ $(-i)$ -linear strand” of a minimal injective resolution of N^\bullet .

[2, 11] treated $d_i(\mathbf{L}(N))$ and $e_i(N)$ for a module $N \in \text{gr } E$ more or less indirectly. But their approach is very different from ours. They use *Cartan (co)homology* of N . See [2, 11] for the definition of this (co)homology. Let $\mathbf{v} = v_1, \dots, v_n$ be a basis of V which is *generic* with respect to N in the sense of [2, Definition 4.7]. As [2, 11], we set $H_i(k) := H_i(v_1, \dots, v_k; N)$ and $H^i(k) := H^i(v_1, \dots, v_k; N)$ to be Cartan (co)homologies. Note that $H_i(n) = \text{Tor}_i^E(K, N)$, $H^i(n) = \text{Ext}_E^i(K, N)$ and $H^i(v_1, \dots, v_k; N^*) \cong H_i(v_1, \dots, v_k; N)^*$. It follows from the argument in §6 of [2] that the function $j \mapsto \dim_K H^{i+j}(k)_j$ is a polynomial function for $j \gg 0$. Moreover, $d_i(\mathbf{L}(N)) \leq 0$ if and only if $H^{i+j}(n)_j = 0$ for $j \gg 0$ if and only if $H^{i+j}(k)_j = 0$ for all $k \leq n$ and $j \gg 0$. [2, Proposition 9.4] can be restated as follows: If $d_i(\mathbf{L}(N)) > 0$, we have

$$(1.2) \quad d_i(\mathbf{L}(N)) = n + 1 - \min\{k \mid H^{i+j}(k)_j \neq 0 \text{ for all } j \gg 0\}.$$

A (cohomological) *distinguished pair* for a module $N \in \text{gr } E$ was introduced in [11, Definition 3.4] (see also [2]). Here we generalize this notion to a complex.

Definition 1.6. Let $N^\bullet \in D^b(\text{gr } E)$. We say $(d, i) \in \mathbb{N} \times \mathbb{Z}$ is a *distinguished pair* for N^\bullet if and only if it is distinguished for $\mathbf{L}(N^\bullet)$ (in the sense of Definition 1.1).

Remark 1.7. By (1.2), we see that (d, i) is a distinguished pair for a module $N \in \text{gr } E$ in the above sense if and only if $(n+1-d, i)$ is a “cohomological distinguished pair” for N in the sense of [11]. (Recall that E is a positively graded ring in [2, 11].) [2] also use the term “distinguished pair”. But this is “homological distinguished pair” of [11], and (d, i) is a distinguished pair for N in our sense if and only if $(n+1-d, n-i)$ is a distinguished pair for N^* in the sense of [2].

Corollary 1.8 (c.f. [11, Theorem 3.8]). *Let $N^\bullet \in D^b(\text{gr } E)$. A pair (d, i) is distinguished for N^\bullet if and only if $(d, 2n-d-i)$ is distinguished for $\mathbf{D}_E(N^\bullet)$. If this is the case, we have $e_i(N^\bullet) = e_{2n-d-i}(\mathbf{D}_E(N^\bullet))$.*

Proof. For the first statement, it suffices to prove the direction \Rightarrow . By Theorem 1.2, $(d, -d-i)$ is a distinguished pair for $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2n)[2n]$. For a complex $M^\bullet \in D^b(\text{gr } S)$, we have $H^j(M^\bullet(-2n)[2n]) = H^{2n+j}(M^\bullet)(-2n)$ and $d_j(M^\bullet(-2n)[2n]) = d_{2n+j}(M^\bullet)$. Thus $(d, 2n-d-i)$ is distinguished for $\mathbf{L} \circ \mathbf{D}_E(N^\bullet)$. The last equality follows from Theorem 1.2 (2). \square

For a module $N \in \text{gr } E$, $d_i(\mathbf{L}(N))$ can be 0 quite often. But we have the following.

Proposition 1.9. *Assume that a module $N \in \text{gr } E$ does not have a free summand. If (d, i) is a distinguished pair for N , then we have $d > 0$.*

Proof. Let $0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ (resp. $\dots \rightarrow I^{-1} \rightarrow I^0 \rightarrow N \rightarrow 0$) be a minimal injective (resp. projective) resolution of N . For $j \geq 0$, set $\Omega_j(N) := (\ker(I^j \rightarrow I^{j+1}))[-j]$. Obviously, $0 \rightarrow \Omega_j(N) \rightarrow I^j \rightarrow I^{j+1} \rightarrow \dots$ is a minimal injective resolution. On the other hand, since N does not have a free summand, $\dots \rightarrow I^{-1} \rightarrow I^0 \rightarrow \dots \rightarrow I^{j-1} \rightarrow \Omega_j(N) \rightarrow 0$ is a minimal projective resolution. If $d_i(\mathbf{L}(N)) > 0$, then $d_i(\mathbf{L}(\Omega_j(N))) = d_i(\mathbf{L}(N))$ for all $j \geq 0$. But, if $d_i(\mathbf{L}(N)) = 0$, then $d_i(\mathbf{L}(\Omega_j(N))) = -\infty$ for $j \gg 0$. On the other hand, since a minimal injective resolution of N^* is the dual of a minimal projective resolution of N , we have $d_i(\mathbf{L}(N^*)) = d_i(\mathbf{L}(\Omega_j(N)^*))$ for all i and all $j \geq 0$. So N^* and $\Omega_j(N)^*$ have the same distinguished pairs. For a contradiction, we assume that $(0, i)$ is a distinguished pair for N . Then $(0, 2n-i)$ is a distinguished pair for N^* and $\Omega_j(N)^*$. So $(0, i)$ is a distinguished pair for $\Omega_j(N)$ for all $j \geq 0$. This contradicts the above observation. \square

We say a distinguished pair (d, i) is *positive*, if $d > 0$. Since [2, 11] study a distinguished pair for a module, they only treat a positive one.

Remark 1.10. When N^\bullet is a module, Corollary 1.8 was proved in [11, Theorem 3.8]. On the other hand, for positive distinguished pairs, we can prove the corollary from [11, Theorem 3.8] directly: Let I^\bullet be an injective resolution of N^\bullet and P^\bullet a projective resolution of I^\bullet . From the quasi-isomorphism $f : P^\bullet \rightarrow I^\bullet$, we have the exact complex $(T^\bullet, \partial^\bullet) := \text{cone}(f)$. Then $N = \ker \partial_0$ (resp. N^*) has the same positive distinguished pairs as N^\bullet (resp. $\mathbf{D}_E(N^\bullet)$).

A variant of BGG correspondence gives an equivalence $\underline{\mathrm{gr}} E \cong D^b(\mathrm{Coh}(\mathbb{P}^{n-1}))$ of triangulated categories, where $\underline{\mathrm{gr}} E$ is the stable category, and $\mathrm{Coh}(\mathbb{P}^{n-1})$ is the category of coherent sheaves on $\mathbb{P}^{n-1} = \mathrm{Proj} S$. More precisely, the composition of the functor $\mathbf{L} : \mathrm{gr} E \rightarrow D^b(\mathrm{gr} S)$ and the natural functor $D^b(\mathrm{gr} S) \rightarrow D^b(\mathrm{Coh}(\mathbb{P}^{n-1}))$ induces this equivalence. Note that the functor $\mathrm{gr} S \ni M \rightarrow \tilde{M} \in \mathrm{Coh}(\mathbb{P}^{n-1})$ ignores modules of finite length. Hence if $d_i(M^\bullet) = 0$ then $H^i(\tilde{M}^\bullet) = 0$. In this sense, the duality in [11] corresponds to a duality on $D^b(\mathrm{Coh}(\mathbb{P}^{n-1}))$.

In the rest of this section, we assume that K is algebraically closed. Let $N \in \mathrm{gr} E$. Following [1], we say $v \in E_{-1} = V$ is N -regular if $\mathrm{Ann}_N(v) = vN$. It is easy to see that v is N -regular if and only if it is N^* -regular. We say $V_E(N) = \{v \in V \mid v \text{ is not } N\text{-regular}\}$ is the *rank variety* of N (see [1]). [1, Theorem 3.1] states that $V_E(N)$ is an algebraic subset of $V = \mathrm{Spec} S$, and $\dim V_E(N) = \max\{d_i(\mathbf{L}(N)) \mid i \in \mathbb{Z}\}$. By the above remark, $V_E(N) = V_E(N^*)$. We can refine this observation using the grading of N .

Recall that S can be seen as the Yoneda algebra $\mathrm{Ext}_E^*(K, K)$, and $\mathrm{Ext}_E^*(K, N)$ has the S -module structure. By the same argument as [1, Theorem 3.9] (see also the proof of [7, Corollary 3.2 (b)]), we have that

$$V_E(N) = \{v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \mathrm{Ann}_S(\mathrm{Ext}_E^*(K, N))\}.$$

But $[\mathrm{Ext}_E^{*+i}(K, N)]_* := \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}_E^{j+i}(K, N)_j$ is an S -module which is isomorphic to $H^i(\mathbf{L}(N))$ (see the proof of [6, Proposition 2.3]), and we have $\mathrm{Ext}_E^*(K, N) \cong \bigoplus_{j \in \mathbb{Z}} [\mathrm{Ext}_E^{*+i}(K, N)]_*$. Set

$$V_E^i(N) = \{v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \mathrm{Ann}_S([\mathrm{Ext}_E^{*+i}(K, N)]_*)\}.$$

We have $V_E(N) = \bigcup_i V_E^i(N)$ and $d_i(\mathbf{L}(N)) = \dim V_E^i(N)$. For an algebraic set $X \subset \mathrm{Spec} S$ of dimension d , set $\mathrm{Top}(X)$ to be the union of the all irreducible components of X of dimensions d .

Proposition 1.11. *If (d, i) is a distinguished pair for $N \in \mathrm{gr} E$, then we have $\mathrm{Top}(V_E^i(N)) = \mathrm{Top}(V_E^{2n-d-i}(N^*))$.*

Proof. By the proof of Theorem 1.2, $\mathrm{Ann}_S(H^i(\mathbf{L}(N)))$ has the same top dimensional components as $\mathrm{Ann}_S(H^{-d-i}(\mathbf{D}_S \circ \mathbf{L}(N)))$. \square

In the above situation, we have $V_E^i(N) \neq V_E^{2n-d-i}(N^*)$ in general.

2. SQUAREFREE CASE

In this section, we regard $S = K[x_1, \dots, x_n]$ as an \mathbb{N}^n -graded ring with $\deg x_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 is in the i th position. Similarly, $E = K\langle y_1, \dots, y_n \rangle$ is a $-\mathbb{N}^n$ -graded ring with $\deg y_i = -\deg x_i$. Let $^*\mathrm{gr} S$ (resp. $^*\mathrm{gr} E$) be the category of finitely generated \mathbb{Z}^n -graded S -modules (resp. E -modules). The functors \mathbf{L} and \mathbf{R} defining the BGG correspondence $D^b(\mathrm{gr} S) \cong D^b(\mathrm{gr} E)$ also work in the \mathbb{Z}^n -graded context. That is, the functors $\mathbf{L} : D^b(^*\mathrm{gr} E) \rightarrow D^b(^*\mathrm{gr} S)$ and $\mathbf{R} : D^b(^*\mathrm{gr} S) \rightarrow D^b(^*\mathrm{gr} E)$ are defined by the same way as the \mathbb{Z} -graded case, and they give an equivalence $D^b(^*\mathrm{gr} S) \cong D^b(^*\mathrm{gr} E)$, see [13, Theorem 4.1]. Note that the dualizing

complex ω^\bullet of S is \mathbb{Z}^n -graded, and $\mathbf{D}_S(-) = \text{Hom}_S^\bullet(-, \omega^\bullet)$ is also a duality functor on $D^b(*\text{gr } S)$. Similarly, $\mathbf{D}_E(-) = \text{Hom}_E(-, E)$ is a duality functor on $D^b(*\text{gr } E)$. As Proposition 1.5, for $N^\bullet \in D^b(*\text{gr } E)$, we have $\mathbf{D}_S \circ \mathbf{L}(N^\bullet) \cong \mathbf{L} \circ \mathbf{D}_E(N^\bullet)(-2)[2n]$ in $D^b(*\text{gr } S)$. Here we set $\mathbf{j} := (j, j, \dots, j) \in \mathbb{N}^n$ for $j \in \mathbb{Z}$.

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, set $\text{supp}(\mathbf{a}) := \{i \mid a_i > 0\} \subset [n] := \{1, \dots, n\}$ and $|\mathbf{a}| = \sum_{i=1}^n a_i$. We say $\mathbf{a} \in \mathbb{Z}^n$ is *squarefree* if $a_i = 0, 1$ for all $i \in [n]$. When $\mathbf{a} \in \mathbb{Z}^n$ is squarefree, we sometimes identify \mathbf{a} with $\text{supp}(\mathbf{a})$.

Definition 2.1 ([12]). We say a \mathbb{Z}^n -graded S -module M is *squarefree*, if the following conditions are satisfied.

- (a) M is \mathbb{N}^n -graded (i.e., $M_{\mathbf{a}} = 0$ if $\mathbf{a} \notin \mathbb{N}^n$) and finitely generated.
- (b) The multiplication map $M_{\mathbf{a}} \ni y \mapsto (\prod x_i^{b_i}) \cdot y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a} + \mathbf{b}) = \text{supp}(\mathbf{a})$.

For a simplicial complex $\Delta \subset 2^{[n]}$, the Stanley-Reisner ideal $I_\Delta := (\prod_{i \in F} x_i \mid F \notin \Delta)$ and the Stanley-Reisner ring $K[\Delta] := S/I_\Delta$ are squarefree modules. Note that if M is squarefree then $M_{\mathbf{a}} \cong M_F$ as K -vector spaces for all $\mathbf{a} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{a}) = F$. Let Sq_S be the full subcategory of $*\text{gr } S$ consisting of squarefree modules. In $*\text{gr } S$, Sq_S is closed under kernels, cokernels and extensions ([12, Lemma 2.3]), and we have that $D^b(\text{Sq}_S) \cong D_{\text{Sq}_S}^b(*\text{gr } S)$. If $M^\bullet \in D^b(\text{Sq}_S)$, then $\mathbf{D}_S(M^\bullet) \in D_{\text{Sq}_S}^b(*\text{gr } S)$ (see [13]). So \mathbf{D}_S gives a duality functor on $D^b(\text{Sq}_S)$.

Definition 2.2 (Römer [11]). A \mathbb{Z}^n -graded E -module $N = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} N_{\mathbf{a}}$ is *squarefree* if N is finitely generated and $N = \bigoplus_{F \subset [n]} N_{-F}$.

For example, any monomial ideal of E is a squarefree module. Any monomial ideal of E is of the form $J_\Delta = (\prod_{i \in F} y_i \mid F \notin \Delta)$ for some simplicial complex $\Delta \subset 2^{[n]}$. We say $K\{\Delta\} := E/J_\Delta$ is the *exterior face ring* of Δ .

Let Sq_E be the full subcategory of $*\text{gr } E$ consisting of squarefree E -modules. Then there exist functors $\mathcal{S} : \text{Sq}_E \rightarrow \text{Sq}_S$ and $\mathcal{E} : \text{Sq}_S \rightarrow \text{Sq}_E$ giving an equivalence $\text{Sq}_S \cong \text{Sq}_E$. Here $\mathcal{S}(N)_F = N_{-F}$ for $N \in \text{Sq}_E$, and the multiplication map $\mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F \cup \{i\}}$ for $i \notin F$ is given by $\mathcal{S}(N)_F = N_{-F} \ni z \mapsto (-1)^{\alpha(i, F)} y_i z \in N_{-(F \cup \{i\})} = \mathcal{S}(N)_{F \cup \{i\}}$, where $\alpha(i, F) = \#\{j \in F \mid j < i\}$. For example, $\mathcal{S}(K\{\Delta\}) = K[\Delta]$. See [11] for further information. Of course, \mathcal{S} and \mathcal{E} can be extended to the functors between $D^b(\text{Sq}_S)$ and $D^b(\text{Sq}_E)$.

If $N \in \text{Sq}_E$, then $N^* = \text{Hom}_E(N, E)$ is squarefree again. So $(-)^*$ gives the duality functor \mathbf{D}_E on $D^b(\text{Sq}_E)$. For example, $K\{\Delta\}^* = J_{\Delta^\vee}$, where $\Delta^\vee = \{F \subset [n] \mid [n] \setminus F \notin \Delta\}$ is the Alexander dual complex of Δ . We have the *Alexander duality functor* $\mathbf{A} := \mathcal{S} \circ \mathbf{D}_E \circ \mathcal{E}$ on Sq_S (or $D^b(\text{Sq}_S)$). Of course, $\mathbf{A}(K[\Delta]) = I_{\Delta^\vee}$. In general, we have $\mathbf{A}(H^i(M^\bullet))_F = (H^{-i}(M^\bullet)_{[n] \setminus F})^\vee$.

An associated prime ideal of $M \in *\text{gr } S$ is of the form $P_F = (x_i \mid i \notin F)$ for some $F \subset [n]$. Let $M \in \text{Sq}_S$ be a squarefree module. A monomial prime ideal P_F is a minimal prime of M if and only if F is a maximal element of the set $\{G \subset [n] \mid M_G \neq 0\}$. The following is a squarefree version of Definition 1.1.

Definition 2.3. We say $(F, i) \in 2^{[n]} \times \mathbb{Z}$ is a *distinguished pair* for a complex $M^\bullet \in D^b(\text{Sq}_S)$, if P_F is a minimal prime of $H^i(M^\bullet)$ and $H^j(M^\bullet)_G = 0$ for all j with $j < i$ and $G \supset F$ with $|G| < |F| + i - j$.

Theorem 2.4. Let $M^\bullet \in D^b(\text{Sq}_S)$. A pair (F, i) is distinguished for M^\bullet if and only if $(F, -|F| - i)$ is distinguished for $\mathbf{D}_S(M^\bullet)$. If this is the case, $\dim_K H^i(M^\bullet)_F = \dim_K H^{-|F|-i}(\mathbf{D}_S(M^\bullet))_F$.

Proof. Like the proof of Theorem 1.2, we consider the spectral sequence $E_2^{p,q} = \text{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet) \Rightarrow \text{Ext}_S^{p+q}(M^\bullet, \omega^\bullet)$. Then $E_r^{p,q}$ is squarefree for all p, q and $r \geq 2$. When we consider a distinguished pair (F, i) , we set

$$\dim_F M := \begin{cases} -\infty & \text{if } M_G = 0 \text{ for all } G \supset F \\ \max\{|G| \mid G \supset F, M_G \neq 0\} & \text{otherwise} \end{cases}$$

for $M \in \text{Sq}_S$. Set $d_i(M^\bullet) := \dim_F H^i(M^\bullet)$ and $e_2^{p,q} := \dim_F \text{Ext}_S^p(H^{-q}(M^\bullet), \omega^\bullet)$ for $M^\bullet \in D^b(\text{Sq}_S)$. We also remark that $\dim_K M_F = l_{S_{P_F}}(M \otimes_S S_{P_F})$ for $M \in \text{Sq}_S$. The equation (1.1) holds in this context, and the proof of Theorem 1.2 works verbatim. \square

If $N^\bullet \in D^b(\text{Sq}_E)$, then it is easy to see that $\mathbf{L}(N^\bullet)(-1) \in D^b(\text{Sq}_S)$. So $\mathcal{L}(-) := \mathbf{L}(-)(-1)$ gives a functor from $D^b(\text{Sq}_E)$ to $D^b(\text{Sq}_S)$. Moreover, we have $\mathcal{L} \cong \mathbf{A} \circ \mathbf{D}_S \circ \mathcal{S}$ by [13, Proposition 4.3].

Definition 2.5. Let $N^\bullet \in D^b(\text{Sq}_E)$. We say (F, i) is a *distinguished pair* for N^\bullet if it is a distinguished pair for $\mathcal{L}(N^\bullet) \in D^b(\text{Sq}_S)$ in the sense of Definition 2.3.

The next result can be proved by the same way as Corollary 1.8 using Theorem 2.4.

Proposition 2.6. Let $N^\bullet \in D^b(\text{Sq}_E)$. A pair (F, i) is distinguished for N^\bullet if and only if $(F, 2n - |F| - i)$ is distinguished for $\mathbf{D}_E(N^\bullet)$. If this is the case, we have $\dim_K H^i(\mathcal{L}(N^\bullet))_F = \dim_K H^{2n-|F|-i}(\mathcal{L} \circ \mathbf{D}_E(N^\bullet))_F$.

If $M^\bullet \in D^b(*_{\text{gr}} S)$, then $\text{Tor}_i^S(K, M^\bullet) := H^{-i}(K \otimes_E P^\bullet)$ is a \mathbb{Z}^n -graded module, where P^\bullet is a graded free resolution of M^\bullet . Set $\beta_{i,\mathbf{a}}(M^\bullet) := \dim_K \text{Tor}_i^S(K, M^\bullet)_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^n$. We say $\beta_{i,\mathbf{a}}(M^\bullet)$ is the (i, \mathbf{a}) th Betti number of M^\bullet . If $M^\bullet \in D^b(\text{Sq}_S)$ and $\beta_{i,\mathbf{a}}(M^\bullet) \neq 0$, then \mathbf{a} is squarefree (see [13]).

Definition 2.7 (c.f. [3]). A Betti number $\beta_{i,F}(M^\bullet) \neq 0$ is *extremal* if $\beta_{j,G}(M^\bullet) = 0$ for all $(j, G) \neq (i, F)$ with $j \geq i$, $G \supset F$, and $|G| - j > |F| - i$.

Some of known results and backgrounds of extremal Betti numbers are found in the introduction of the present paper.

Proposition 2.8 (c.f. [2]). Let $M^\bullet \in D^b(\text{Sq}_S)$ and $N^\bullet := \mathcal{E}(M^\bullet) \in D^b(\text{Sq}_E)$. A pair (F, i) is distinguished for $\mathbf{D}_E(N^\bullet)$ if and only if $\beta_{i+|F|-n,F}(M^\bullet)$ is an extremal Betti number. If this is the case, then $\beta_{i+|F|-n,F}(M^\bullet) = \dim_K H^i(\mathcal{L} \circ \mathbf{D}_E(N^\bullet))_F$.

Proof. For $j \in \mathbb{Z}$ and $G \subset [n]$, we have the following.

$$\begin{aligned}
\beta_{j,G}(M^\bullet) &= \dim_K[H^{|G|-j-n}(\mathbf{D}_S \circ \mathbf{A}(M^\bullet))]_{[n] \setminus G} \quad (\text{by [13, Corollary 3.6]}) \\
&= \dim_K[H^{n+j-|G|}(\mathbf{A} \circ \mathbf{D}_S \circ \mathbf{A}(M^\bullet))]_G \\
&= \dim_K[H^{n+j-|G|}(\mathcal{L} \circ \mathcal{E} \circ \mathbf{A}(M^\bullet))]_G \\
&= \dim_K[H^{n+j-|G|}(\mathcal{L} \circ \mathbf{D}_E(N^\bullet))]_G.
\end{aligned}$$

The assertion easily follows from this equality. \square

Corollary 2.9. *Let $M^\bullet \in D^b(\text{Sq}_S)$. A Betti number $\beta_{i,F}(M^\bullet)$ is extremal if and only if so is $\beta_{|F|-i,F}(\mathbf{A}(M^\bullet))$. If this is the case, $\beta_{i,F}(M^\bullet) = \beta_{|F|-i,F}(\mathbf{A}(M^\bullet))$.*

Proof. If $\beta_{i,F}(M^\bullet)$ is extremal, then $(F, n+i-|F|)$ is a distinguished pair for $\mathbf{D}_E \circ \mathcal{E}(M^\bullet)$ by Proposition 2.8. By Proposition 2.6, $(F, n-i)$ is a distinguished pair for $\mathcal{E}(M^\bullet) \cong \mathbf{D}_E \circ \mathcal{E} \circ \mathbf{A}(M^\bullet)$. So $\beta_{|F|-i,F}(\mathbf{A}(M^\bullet))$ is extremal. The converse implication can be proved by the same way. The last equality follows from Proposition 2.6. \square

This corollary generalizes results of Bayer-Charalambous-Popescu [3], Römer [11] and Miller [10]. Roughly speaking, the above proof is a “complex version” of [11]. But, his argument itself does not work in the \mathbb{Z}^n -graded context, since he use a generic base change of $V = E_{-1}$.

For $M^\bullet \in D^b(\text{Sq}_S)$. Set $\text{proj. dim}(M^\bullet) = \max\{i \mid \beta_{i,F}(M^\bullet) \neq 0 \text{ for some } F\}$ and $\text{reg}(M^\bullet) = \max\{|F|-i \mid \beta_{i,F}(M^\bullet) \neq 0\}$. Since Betti numbers $\beta_{i,F}(M^\bullet)$ which give $\text{proj. dim}(M^\bullet)$ or $\text{reg}(M^\bullet)$ are extremal, the next result follows from Corollary 2.9.

Corollary 2.10 (c.f.[10, 11]). *If $M^\bullet \in D^b(\text{Sq}_S)$, then $\text{proj. dim}(M^\bullet) = \text{reg}(\mathbf{A}(M^\bullet))$.*

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